Backreaction of excitations on a vortex *

by

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Abstract

Excitations of a vortex are usually considered in a linear approximation neglecting their backreaction on the vortex. In the present paper we investigate backreaction of Proca type excitations on a straightlinear vortex in the Abelian Higgs model. We propose exact Ansatz for fields of the excited vortex. From initial set of six nonlinear field equations we obtain (in a limit of weak excitations) two linear wave equations for the backreaction corrections. Their approximate solutions are found in the cases of plane wave and wave packet type excitations. We find that the excited vortex radiates vector field and that the Higgs field has a very broad oscillating component.

TPJU 10/96
May 1996
hep-th/9607116

*Work supported in part by grant KBN 2 P302 049 05.

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1 INTRODUCTION

String–like vortices play essential role in several branches of physics, e.g., in condensed matter physics, in physics of hadrons, or in field-theoretical cosmology. The vortices have very interesting, intricate dynamics. Its understanding is crucial for extracting predictions from many theoretical models.

We are interested in a particular aspect of vortex dynamics – properties of excited vortex. Excitations of vortices have been studied already for some time, see e.g. [1] for recent results and references to older papers. Our interest in this aspect of vortex dynamics stems from the fact that there is no large energy barrier or exclusion principle for production of excited vortices. Therefore, one may expect that in formation or interaction processes excited vortices will occur quite often, similarly as excited atoms or molecules are obtained in a recombination process. Such expectations are supported by computer simulations of the intercommutation of vortices [2], where one observes that parts of vortices which are close to the interaction point are locally excited. For concreteness, we consider the Abrikosov-Nielsen-Olesen (ANO) vortex [3] in the Abelian Higgs model with a particular type of axially symmetric excitation discussed already in [4], [5]. The excitation can be described as a bound state of longitudinally polarized vector particles with the vortex. In the present paper we study the excited vortex from a general field-theoretical viewpoint – applications to vortices in superconductors will be presented in a forthcoming paper [6].

The problem we are addressing ourselves to can be described as follows. The excitation is usually obtained as a first order correction (in an expansion with respect to amplitude of the excitation) to the unexcited ANO vortex fields, which are taken as the zeroth order fields. Up to the first order approximation the total field configuration is just a sum of the fields of unexcited vortex and of the excitation. However, because of nonlinearity of the pertinent field equations there are higher order corrections to this sum. Generally, they can be regarded as giving interaction of the excitation with the unexcited ANO vortex, and also selfinteraction of the excitation. Such second and higher order effects in the case of vortices seem to be unexplored as yet. In the present paper we calculate second order corrections in the perturbative expansion with respect to amplitude of the excitation in order to check what kind of physical effects appear. It turns out that these second order effects are quite interesting. We find that the excited vortex emits waves of the vector field, and that it has larger and oscillating width. All corrections found in the second order can be regarded as a backreaction of the excitation on the ANO vortex. Selfinteraction of the excitation will appear in the third order.

The present paper is based on our earlier work [7] in which we have obtained approximate analytic formulae for the ANO vortex, as well as for the excitation profile and frequency (F^{ANO}, χ^{ANO}, $\alpha(r)$ and ω_0 below). We have also considered there the backreaction in the case of homogeneously excited vortex, which is a particularly simple case because of the lack of dependence on position along the vortex. Certain results obtained in [7] hold for excitations of finite amplitude, e.g., we have found an upper bound on

amplitude of the homogeneous excitation. In the present paper we try to cope with inhomogeneous excitations. The price for greater generality is that we can investigate only excitations which have small amplitude – our approach is perturbative one.

In order to keep technical details at a reasonable level we consider only elementary vortex with unit topological charge. The vortex is straight-linear, axially symmetric, and its core lies on the x^3- axis. Another simplification we adopt in our paper is that the parameter $\kappa \equiv \sqrt{2q^2/\lambda}$ is assumed to be very small, $\kappa \ll 1/2$. In a static case this corresponds to strongly II type superconductors. Actually, in several places we consider the limiting case $\kappa \to 0$, but in fact we expect that our conclusion remain qualitatively valid for all $\kappa < 1/2$. The point $\kappa = 1/2$ is distinguished by a change of asymptotic form of the Higgs field [8] – a polynomial approximation used in [7] has not been extended as yet to this and higher values of κ.

The plan of our paper is as follows. In the next Section we present Ansatz and field equations for the axially symmetric excited vortex. In Section 3 we describe the perturbative expansion with respect to the amplitude of excitation. Sections 4 and 5 are devoted to detailed analysis of the backreaction in the cases of plane wave and wave packet type excitations. In Section 6 we have collected several remarks.

2 THE ANSATZ

Our notation is fixed by writing the Lagrangian of the Abelian Higgs model in the following form,

$$\mathcal{L} = (D_\mu\phi)^* (D^\mu\phi) - \frac{\lambda}{4}\left(\phi^*\phi - \frac{2m^2}{\lambda}\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1}$$

where

$$D_\mu\phi = \partial_\mu\phi + iqA_\mu\phi, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Signature of the space–time metric is $(+,-,-,-)$ and $c = 1$ as usual. Masses of the scalar and vector particles are equal to $m_H = \sqrt{2}m$ and $m_A = \kappa m_H$, respectively. In the following we put $q = 1$.

To describe the excited vortex we use the most general axially symmetric extension of the Abrikosov – Nielsen – Olesen (ANO) Ansatz,

$$\phi = \sqrt{\frac{2m^2}{\lambda}} \exp\Big[i(\theta + \vartheta(\xi^\alpha, r))\Big] F(\xi^\alpha, r), \tag{2}$$

$$A^1 = -m_H \frac{r^2}{r}\frac{1-\chi(\xi^\alpha, r)}{r}, \qquad A^2 = m_H \frac{r^1}{r}\frac{1-\chi(\xi^\alpha, r)}{r}, \tag{3}$$

$$A^\beta = m_H A^\beta(\xi^\alpha, r). \tag{4}$$

Here we use the dimensionless variables,

$$r^k = m_H x^k, \qquad r = \sqrt{(r^1)^2 + (r^2)^2};$$

$$\xi^\alpha = m_H x^\alpha.$$

The indices α, β take values 0, 3, while $i, k = 1, 2$. θ denotes the azimuthal angle in the (r^1, r^2) plane. The functions F, χ, A^β and ϑ are dimensionless. For the unexcited, static ANO vortex

$$\vartheta = 0, \qquad A^\beta = 0, \tag{5}$$

and F, χ do not depend on ξ^α.

The Ansatz (2–4) and Euler–Lagrange equations obtained from Lagrangian (1) lead to the following equations for F, ϑ, A^β and χ,

$$-\partial_\beta\partial^\beta F + F'' + \frac{1}{r}F' + \left[(A_\beta + \partial_\beta\vartheta)^2 + \frac{1}{2}(1 - F^2) - \left(\frac{\chi^2}{r^2} + \vartheta'^2\right)\right] F = 0, \tag{6}$$

$$F\partial_\beta\partial^\beta\vartheta + 2\partial_\beta F\partial^\beta\vartheta - 2F'\vartheta' - F\left(\vartheta'' + \frac{1}{r}\vartheta'\right) + 2A^\beta\partial_\beta F + \left(\partial_\beta A^\beta\right) F = 0, \tag{7}$$

$$-\partial_\alpha\partial^\alpha A^\beta + \partial^\beta\left(\partial_\alpha A^\alpha\right) + \left(A^{\beta\prime\prime} + \frac{A^{\beta\prime}}{r}\right) - \kappa^2 F^2 A^\beta = \kappa^2 F^2 \partial^\beta\vartheta, \tag{8}$$

$$-\partial_\beta\partial^\beta\chi + \chi'' - \frac{\chi'}{r} = \kappa^2 F^2 \chi, \tag{9}$$

$$\partial_\beta A^{\beta\prime} = \kappa^2 F^2 \vartheta'. \tag{10}$$

Here ' stands for d/dr. Equations for the vector field we have split into the $\beta = 0, 3$ and $i = 1, 2$ components.

Equation (7) has the form of wave equation for ϑ. It turns out that it may be omitted because it follows from Eqs. (8), (10). To check this, act with ∂_β on both sides of Eq. (8) to obtain

$$\partial_\beta A^{\beta\prime\prime} = -\frac{1}{r}\partial_\beta A^{\beta\prime} + 2\kappa^2 F\partial_\beta F\left(\partial^\beta\vartheta + A^\beta\right) + \kappa^2 F^2\left(\partial_\beta\partial^\beta\vartheta + \partial_\beta A^\beta\right).$$

Next, eliminate $\partial_\beta A^{\beta\prime}$ from this formula using the relation (10) and divide the resulting formula by $\kappa^2 F$. The result coincides with Eq. (7).

Thus, in addition to Eqs. (6), (9) which are nontrivial also for the unexcited ANO vortex (cf. formulae (5)), we have two nonlinear wave equations more, namely the two components of Eq. (8). The phase field ϑ is determined from Eq. (10):

$$\vartheta(\xi, r) = -\kappa^{-2}\int_r^\infty ds\; F^{-2}(\xi, s)\partial_\beta A'^\beta(\xi, s). \tag{11}$$

This formula implies a gauge fixing for a residual gauge freedom present in the Ansatz (2-4) and consisting of gauge transformations with gauge function $\delta\vartheta$ dependent only

on ξ^α. Using this freedom we may adjust ϑ so that it vanishes for $r \to \infty$ – this is the gauge implied by formula (11). Notice that in spite of U(1) gauge invariance of Lagrangian (1) the phase $\vartheta(\xi^\alpha, r)$ cannot be gauged away within the Ansatz (2-4) because the corresponding gauge transformation would introduce terms $\delta A^i \sim r^i/r$ which are not compatible with formulae (3) for A^i. In an alternative form of the Ansatz we could have put $\vartheta = 0$ and introduced the terms $\sim r^i/r$ in formulae (3).

Apart from the wave equations, we also require that the functions F, χ, A^β obey a number of boundary conditions. For F and χ at $r = 0$ we take the standard ANO vortex conditions

$$F(0) = 0, \qquad \chi(0) = 1, \tag{12}$$

which follow from the requirement that the fields ϕ and A^i are regular at $r = 0$ – we are not interested here in singular solutions. Boundary conditions at $r \to \infty$ are less obvious. First, in our perturbative approach the pair of equations (6, 9) is replaced by infinite set of equations, a pair in each order. It turns out that it is not always possible to impose the conditions that the perturbative contributions to F and χ exponentially vanish for $r \to \infty$. The reason is that most of the pertubatively obtained equations have radiation type solutions which vanish very slowly or do not vanish at all for large r. In this case we adopt the Helmholtz condition which states that for $r \to \infty$ only outgoing radiation waves are present, [9].

As for the functions A^β, we shall require that they are regular for all $r \geq 0$ and that they vanish for $r \to \infty$. These conditions correspond to the physical picture that the excitation is localised on the vortex, i.e. that it can be regarded as a bound state of the $\beta = 0, 3$ components of the vector field A_μ with the vortex.

3 THE PERTURBATIVE EXPANSION

The set of nonlinear wave equations (6), (8), (9) is very complicated. In the following part of our paper we attempt to construct its perturbative solution relevant for the excited vortex in the case when the excitation, represented by the A^β fields, has a small amplitude which will be regarded as expansion parameter. The ensuing picture is as follows. In the leading order (ε^0) we have the unperturbed ANO vortex. In the order ε^1 waves of the A^β field travelling along the vortex appear. They can be described as a 2–dimensional free Proca field living on the vortex. In the next order (ε^2) backreaction of the Proca wave on the vortex appears. It introduces two new features: perturbation of radius of the vortex and radiation from the vortex.

The perturbative expansion has the form

$$A_\alpha = \varepsilon A^{(1)}_\alpha + \mathcal{O}(\varepsilon^3), \tag{13}$$

$$F = F^{\mathrm{ANO}} + \varepsilon^2 F^{(2)} + \mathcal{O}(\varepsilon^4), \quad \chi = \chi^{\mathrm{ANO}} + \varepsilon^2 \chi^{(2)} + \mathcal{O}(\varepsilon^4), \tag{14}$$

where $F^{\rm ANO}$, $\chi^{\rm ANO}$ give the well-known static, unexcited ANO vortex. Approximate formulae for $F^{\rm ANO}$, $\chi^{\rm ANO}$ are given below. ε is just an auxilliary book–keeping parameter — it is put to 1 at the end of calculations. The power expansion in ε is what we mean by the expansion in amplitude of the excitation $A^{(1)}_{\alpha}$. The powers of ε in formulae (13, 14) are suggested by the form of Eqs. (6), (8), (9).

Let us first eliminate the phase ϑ. In the leading order, Eq. (10) admits as the solution

$$\vartheta^{(1)} = 0, \tag{15}$$

with

$$\partial^{\beta} A^{(1)}_{\beta} = 0. \tag{16}$$

This solution is compatible with Eq. (8) because $\partial_{\beta} F^{\rm ANO} = 0$.

Now, with (15) taken into account, Eq. (8) in the order ε^1 decouples from the other equations,

$$-\partial_{\alpha}\partial^{\alpha} A^{(1)}_{\beta} + A^{(1)\prime\prime}_{\beta} + \frac{1}{r} A^{(1)\prime}_{\beta} - \kappa^2 (F^{\rm ANO})^2 A^{(1)}_{\beta} = 0. \tag{17}$$

This equation has the form of 4-dimensional wave equation and as such it has plenty of solutions. We are interested in bound state type solutions, i.e. the ones which exponentially vanish for large r. They can be found with the help of separation of variables ξ^{β} from r,

$$A^{(1)}_{\beta} = W_{\beta}(\xi)\alpha(r), \tag{18}$$

where $W_{\beta}(\xi)$ obey the Lorentz condition

$$\partial_{\beta} W^{\beta} = 0, \tag{19}$$

and the 2–dimensional Proca equation

$$\partial_{\alpha}\partial^{\alpha} W_{\beta} + \omega_0^2 W_{\beta} = 0 \tag{20}$$

with the mass ω_0 given below.

It follows from (17–20) that the profile function of the excitation, i.e. $\alpha(r)$, obeys the equation

$$\omega_0^2 \alpha(r) + \alpha''(r) + \frac{\alpha'(r)}{r} - \kappa^2 (F^{\rm ANO})^2 \alpha(r) = 0. \tag{21}$$

This equation has been considered in [5], [7]. Its vanishing for $r \to \infty$ solution is quoted below. Without any loss of generality we may normalize $\alpha(r)$. The solution given below is normalized by the condition

$$\alpha(0) = 1.$$

Proca equation (20) possesses infinitely many solutions. Among them there are travelling plane waves and wave packets – they give rise to the inhomogeneous excitations considered in the present paper. The "breathing" homogeneous vortex considered in [7] is obtained for $W^0 = 0, W^3 = W^3(x^0)$, which is a non–propagating, periodic in time and

x^3-independent Proca field standing on the vortex. Condition (19) implies that there exists a potential ψ for the W^β field,

$$W_\beta = \epsilon_{\beta\alpha}\partial^\alpha\psi,$$

where $\epsilon_{\beta\alpha} = -\epsilon_{\alpha\beta}$ and $\epsilon_{03} = +1$. Proca equation (20) will be satisfied if the potential obeys Klein-Gordon equation

$$\partial_\gamma\partial^\gamma\psi + \omega_0^2\psi = 0. \tag{22}$$

In the following Sections we shall consider the plane wave solution of equation (22), namely

$$\psi = N\sin(E(q)\xi^0 - q\xi^3 + \delta_0), \tag{23}$$

where $E(q) = \sqrt{\omega_0^2 + q^2}$, N is a constant, and δ_0 is a constant phase which we shall put to zero.

Another solution of Eq. (22), also considered in the following Sections, is an approximate solution giving a standing wave packet,

$$\psi \approx N\exp(-\xi_3^2/4\Lambda^2)\sin(\xi^0\omega_0), \tag{24}$$

where Λ and N are constants. Average momentum of this wave packet is equal to zero. We also assume that $\omega_0\Lambda \gg 1$. Then, the wave packet has momentum cutoff at small momentum $k_3 \sim \Lambda^{-1} \ll 1$. In this case we may neglect (for a finite time $0 \le \xi^0 < \Lambda^2$) dispersion of the wave packet, which is of course present in exact wave packet solutions of Eq. (22). Such exact wave packet solutions are easy to obtain, nevertheless we prefer the approximate solution because it has significantly simpler form.

Now, let us turn to the remaining two equations (6) and (9). In the order ε^0 they are solved by the unexcited vortex fields $F^{\mathrm{ANO}}, \chi^{\mathrm{ANO}}$. In the order ε^2 they give the following equations for $\chi^{(2)}, F^{(2)}$:

$$\begin{aligned} -\partial_\beta\partial^\beta F^{(2)} + F^{(2)\prime\prime} + \frac{1}{r}F^{(2)\prime} + \frac{1}{2}F^{(2)}\left[1 - 3(F^{\mathrm{ANO}})^2 - \frac{2}{r^2}(\chi^{\mathrm{ANO}})^2\right] = \\ \frac{2}{r^2}\chi^{\mathrm{ANO}}F^{\mathrm{ANO}}\chi^{(2)} - F^{\mathrm{ANO}}W_\beta W^\beta\alpha^2(r), \end{aligned} \tag{25}$$

$$-\partial_\beta\partial^\beta\chi^{(2)} + \chi^{(2)\prime\prime} - \frac{1}{r}\chi^{(2)\prime} - \kappa^2(F^{\mathrm{ANO}})^2\chi^{(2)} = 2\kappa^2F^{\mathrm{ANO}}\chi^{\mathrm{ANO}}F^{(2)}. \tag{26}$$

The boundary conditions for $F^{(2)}, \chi^{(2)}$ at $r = 0$ follow from conditions (12) and from the fact that $F^{\mathrm{ANO}}, \chi^{\mathrm{ANO}}$ already obey these conditions. Therefore,

$$F^{(2)}(r = 0, \xi^\alpha) = 0, \qquad \chi^{(2)}(r = 0, \xi^\alpha) = 0. \tag{27}$$

We shall see that we may also require that

$$F^{(2)}(r, \xi^\alpha) \to 0 \quad \text{for} \quad r \to \infty. \tag{28}$$

As for $\chi^{(2)}$, we find that Eq. (26) has radiation type solutions, so we adopt the outgoing radiation condition.

The last term on the r.h.s of Eq. (25) is a source term for $F^{(2)}$. Due to its presence $F^{(2)}, \chi^{(2)}$ do not vanish. Hence, the backreaction of the excitation on the vortex is always present (one can easily prove that $W_\beta W^\beta \neq 0$ if $W_\beta \neq 0$).

From a mathematical point of view, Eqs. (25), (26) form a set of 4–dimensional, linear wave equations with the coefficients given by the ANO functions $F^{\rm ANO}, \chi^{\rm ANO}$. Solutions of such equations can be investigated with a help of numerical methods. In the present paper we take another route. We will present approximate analytical solutions which we can obtain in the limiting case $\kappa \rightarrow 0$, i.e. when $0 < \kappa \ll 1/2$ (in practice we mean κ not greater than 0.1). This limit is distinguished by two significant simplifications.

First, because $\kappa < 1/2$ we may use relatively simple approximate formulae for the functions $F^{\rm ANO}, \chi^{\rm ANO}, \alpha(r)$ and for the frequency ω_0 obtained in [7]. In the limiting case $\kappa \ll 1/2$ we obtain from formulae given in [7] that for $r \leq r_0 = 4/\sqrt{3}$

$$\chi^{\rm ANO} \simeq 1, \tag{29}$$

$$F^{\rm ANO} \simeq \frac{3}{2r_0} r - \frac{1}{2r_0^3} r^3, \tag{30}$$

$$\alpha(r) \simeq 1, \tag{31}$$

while for $r \geq r_0$

$$\chi^{\rm ANO} \simeq \kappa r K_1(\kappa r), \tag{32}$$

$$F^{\rm ANO} \simeq 1, \tag{33}$$

$$\alpha(r) \simeq c_1 K_0(k_0 r), \tag{34}$$

where

$$c_1 \simeq \frac{23}{3}\kappa^2, \tag{35}$$

and

$$k_0 \simeq 0.85 \, \exp\left\{-\frac{3}{23}\frac{1}{\kappa^2}\right\}. \tag{36}$$

The frequency ω_0 is given by the formula

$$\omega_0 = \sqrt{\kappa^2 - k_0^2}.$$

$K_0(z)$ and $K_1(z)$ are the modified Hankel functions, [10]. The corresponding functions in the two regions smoothly match each other at $r = r_0$. On the r.h.s. of formulae (29-36) we have neglected terms of the order $\kappa^2 \ln \kappa$ or smaller. More accurate formulae for the

ANO vortex and the excitation can be found in [7]. Notice that k_0 is exceedingly small, and therefore the profile function $\alpha(r)$ is very broad, much broader than χ^{ANO} and F^{ANO} which have the width $\sim 1/\kappa$, $\sim r_0$, respectively.

The second simplification occuring for $\kappa \ll 1/2$ is that the first term on the r.h.s. of Eq. (25) can be neglected. To see this, first notice that this term is finite at $r = 0$ because F^{ANO} and $\chi^{(2)}$ vanish at that point. Now, the r.h.s. of Eq. (26) is of the order κ^2, therefore $\chi^{(2)}$ determined from that equation is expected to vanish for $\kappa \to 0$. Therefore, the discussed term for the very small κ will be negligibly small in comparison with the other term on the r.h.s. of Eq. (25) which stays finite in the limit $\kappa \to 0$. Thus, instead of Eq. (25) we may consider the equation

$$-\partial_\beta \partial^\beta F^{(2)} + F^{(2)\prime\prime} + \frac{1}{r} F^{(2)\prime} + \frac{1}{2} F^{(2)} \left[1 - 3(F^{\mathrm{ANO}})^2 - \frac{2}{r^2}(\chi^{\mathrm{ANO}})^2 \right] = -F^{\mathrm{ANO}} W_\beta W^\beta \alpha^2(r). \tag{37}$$

Due to this latter simplification the set of intercoupled equations (25), (26) is replaced by equations (37) and (26) which can be solved one after the other. $F^{(2)}$ determined from Eq. (37) gives the source term in Eq. (26). In the following two Sections we shall present solutions of these equations in the particular cases of the excitation field W^β given by the plane wave (23) and the wave packet (24).

4 BACKREACTION ON THE HIGGS FIELD

In this Section we shall analyse the backreaction of the excitations on the Higgs field F. Our starting point is the equation (37).

It is convenient to pass to Fourier transforms with respect to ξ^α,

$$F^{(2)}(r, \xi) = \frac{1}{2\pi} \int d\omega dk \; \exp(-i\omega\xi^0 + ik\xi^3) f(r, k, \omega), \tag{38}$$

$$\chi^{(2)}(r, \xi) = \frac{1}{2\pi} \int d\omega dk \exp(-i\omega\xi^0 + ik\xi^3) h(r, k, \omega), \tag{39}$$

and

$$w(k, \omega) = \frac{1}{2\pi} \int d\xi^0 d\xi^3 \; \exp(i\omega\xi^0 - ik\xi^3) W_\beta(\xi) W^\beta(\xi). \tag{40}$$

The Fourier transform of Eq.(37) has the following form

$$f'' + \frac{1}{r} f' - \left[\omega^2 - k^2 - \frac{1}{2} + \frac{3}{2}(F^{\mathrm{ANO}}(r))^2 + \frac{1}{r^2}(\chi^{\mathrm{ANO}}(r))^2 \right] f = -F^{\mathrm{ANO}}(r) \alpha^2(r) w(k, \omega). \tag{41}$$

For the plane wave (23) with $\delta_0 = 0$

$$W_\beta(\xi) W^\beta(\xi) = -N^2 \omega_0^2 \cos^2(E(q)\xi^0 - q\xi^3),$$

and

$$w(k,\omega) = -\pi\omega_0^2 N^2 \left[\delta(\omega)\delta(k) + \frac{1}{2}\delta(\omega - 2E(q))\delta(k-2q) + \frac{1}{2}\delta(\omega + 2E(q))\delta(k+2q)\right]. \tag{42}$$

In the case of the wave packet (24)

$$W^\alpha W_\alpha = w_1(\xi_3) + w_2(\xi_3)\cos(2\omega_0\xi^0), \tag{43}$$

where

$$w_1(\xi_3) = \frac{1}{2}N^2(\frac{\xi_3^2}{4\Lambda^4} - \omega_0^2)\exp(-\frac{\xi_3^2}{2\Lambda^2}), \tag{44}$$

$$w_2(\xi_3) = -\frac{1}{2}N^2(\frac{\xi_3^2}{4\Lambda^4} + \omega_0^2)\exp(-\frac{\xi_3^2}{2\Lambda^2}). \tag{45}$$

Formula (40) gives in this case

$$w(k,\omega) = \sqrt{2\pi}\left[\delta(\omega)\tilde{w_1}(k) + \frac{1}{2}\delta(\omega - 2\omega_0)\tilde{w_2}(k) + \frac{1}{2}\delta(\omega + 2\omega_0)\tilde{w_2}(k)\right], \tag{46}$$

where $\tilde{w}_{1,2}$ denote Fourier transforms of $w_{1,2}(\xi_3)$. We shall not need their explicit form.

It is clear from Eq.(41) that f has the form analoguous to (42), (46), respectively,

$$f = \pi\omega_0^2 N^2 \left[f_0(r)\delta(\omega)\delta(k) + \frac{1}{2}f_+(r)\delta(\omega - 2E(q))\delta(k-2q) \right. \tag{47}$$
$$\left. + \frac{1}{2}f_-(r)\delta(\omega + 2E(q))\delta(k+2q)\right],$$

or

$$f = -\sqrt{2\pi}\left[f_0(r)\delta(\omega)\tilde{w_1}(k) + \frac{1}{2}f_+(r)\delta(\omega - 2\omega_0)\tilde{w_2}(k) + \frac{1}{2}f_-(r)\delta(\omega + 2\omega_0)\tilde{w_2}(k)\right]. \tag{48}$$

The negative and positive frequency components are related by complex conjugation,

$$f_- = f_+^*$$

and f_0 is real valued.

Next, we observe that the expression in square bracket on the l.h.s. of Eq. (41) can be simplified a little bit. Namely, the term $\omega^2 - k^2$ is equal to either 0 or $4\omega_0^2$ in both the plane wave and wave packet cases, and for small κ it is negligibly small in comparison with the other terms in the square bracket – a plot of the function $-1/2 + 3/2(F^{\mathrm{ANO}}(r))^2 + (\chi^{\mathrm{ANO}}(r))^2/r^2$ shows that its minimal value is approximately equal to 0.945. Therefore, instead of Eq. (41) we may consider the following universal equation, common for all frequency components and for the plane wave and wave packet cases,

$$f_u'' + \frac{1}{r}f_u' - \left[-\frac{1}{2} + \frac{3}{2}(F^{\mathrm{ANO}}(r))^2 + \frac{1}{r^2}(\chi^{\mathrm{ANO}}(r))^2\right] f_u = F^{\mathrm{ANO}}(r)\alpha^2(r), \tag{49}$$

where

$$f_u = f_0 \approx f_{\pm}. \tag{50}$$

In order to see the backreaction on the Higgs field we have to solve Eq. (49) for f_u. To this end we shall use the polynomial approximation which has proved useful also in our earlier work [7]. The idea is to find approximate solutions of Eq. (49) separately in the regions $r \leq r_0$ and $r \geq r_0$, and to glue them at $r = r_0$.

In the "outer" region, i.e. for $r \geq r_0 = 4/\sqrt{3}$, we use formulae (32-36). Thus, $F^{\text{ANO}}(r) \simeq 1$, and the functions $(\chi^{\text{ANO}})^2/r^2$ and $\alpha(r)^2$ are smooth and very slowly changing with r. Their derivatives with respect to r are of the order κ and k_0, respectively, hence they are close to zero. Therefore, the following function is a reasonable approximation to the exact solution of Eq. (49) in the outer region

$$f_u = \tilde{f}_u(r) \equiv -\alpha(r)^2 \left(1 + \frac{(\chi^{\text{ANO}}(r))^2}{r^2}\right)^{-1} + f_0 K_0(r), \tag{51}$$

where the last term on the r.h.s. is a general, vanishing for $r \to \infty$, solution of the homogeneous counterpart of Eq. (49). f_0 is a constant to be determined from matching conditions at $r = r_0$.

On the other hand, for $r \leq r_0$ the ANO functions are given by polynomials (29-31), and we seek a polynomial approximation also for f_u,

$$f_u = \underline{f}_u \equiv f_1 r + f_3 r^3 + f_5 r^5. \tag{52}$$

We take the fifth order polynomial for f_u because formula (30) gives F^{ANO} up to the r^3 term. Inclusion of the next term ($\sim r^7$) would make sense if we knew F^{ANO} up to $\sim r^5$ term, as it is clear from comparison of the both sides of Eq. (49). Equation (49) gives the following relations

$$f_3 = \frac{3}{16 r_0} - \frac{1}{16} f_1,$$

$$f_5 = -\frac{1}{48 r_0^3} + \frac{9}{64 r_0^2} f_1 - \frac{1}{48} f_3,$$

where f_1 remains undetermined.

At $r = r_0$ we impose matching conditions. They have the following form

$$\underline{f}_u(r_0) = \tilde{f}_u(r_0), \quad \underline{f}'_u(r_0) = \tilde{f}'_u(r_0).$$

These conditions give a set of linear, algebraic equations for the constants f_0, f_1 – the solution is

$$f_0 \approx 5.19, \quad f_1 \approx -0.36. \tag{53}$$

From formulae (38), (47), (48), (50) we obtain the following final expression for the backreaction on the scalar field

$$F^{(2)} \approx -W^{\alpha} W_{\alpha}(\xi) f_u(r), \tag{54}$$

where $f_u(r)$ is given by (51-53). Formula (54) covers the plane wave and wave packet cases. We see that $F^{(2)}$ introduces modulated along the vortex and oscillating in time very broad component in the scalar field of the vortex. Its range is $\sim (k_0 m_H)^{-1}$.

5 BACKREACTION ON THE VECTOR FIELD

Now let us investigate the backreaction on the vector field. Fourier transform of equation (26) has the following form

$$h'' - \frac{1}{r}h' + \left[\omega^2 - k^2 - \kappa^2 (F^{\mathrm{ANO}}(r))^2\right] h = 2\kappa^2 F^{\mathrm{ANO}}(r)\chi^{\mathrm{ANO}}(r) f(r,k,\omega). \tag{55}$$

We shall consider the plane wave and the wave packet cases separately.

5.1 The plane wave case

In analogy to formula (47) we write

$$\begin{aligned} h = 2\kappa^2\pi\omega_0^2 N^2 \Big[& h_0(r)\delta(\omega)\delta(k) + \frac{1}{2}h_+(r)\delta(\omega - 2E(q))\delta(k-2q) \\ & + \frac{1}{2}h_-(r)\delta(\omega + 2E(q))\delta(k+2q)\Big]. \end{aligned} \tag{56}$$

Equation (55) is equivalent to the following set of equations (one equation for each frequency component),

$$h_a'' - \frac{h_a'}{r} + \left[\Omega_a^2 - \kappa^2 (F^{\mathrm{ANO}}(r))^2\right] h_a = F^{\mathrm{ANO}}(r)\chi^{\mathrm{ANO}}(r) f_u(r), \tag{57}$$

where the index a takes values $0, +, -$, and $\Omega_0 = 0, \quad \Omega_\pm = \pm 2\omega_0$. Very important difference with the case of Higgs backreaction f is that in these equations the $4\omega_0^2$ term is of the same order of magnitude as the $\kappa^2 (F^{\mathrm{ANO}})^2$ term (recall that $\omega_0^2 \approx \kappa^2$).

There are two essentially different cases: $\Omega_0^2 = 0$ and $\Omega_\pm^2 = 4\omega_0^2$. In the first case, in the outer region ($r \geq r_0$) Eq. (57) has the form

$$\tilde{h}_0'' - \frac{1}{r}\tilde{h}_0' - \kappa^2 \tilde{h}_0 = \chi^{\mathrm{ANO}}(r)\tilde{f}_u(r), \tag{58}$$

where χ^{ANO} is given by formula (32). Applying standard methods, see e.g. [11], we find the following exact, vanishing at $r \to \infty$, solution

$$\tilde{h}_0(r) = r g_1(r) I_1(\kappa r) + r g_2(r) K_1(\kappa r) + b_0 \kappa r K_1(\kappa r), \tag{59}$$

where

$$g_1(r) = -\int_r^\infty dr'\, \chi^{\mathrm{ANO}}(r') K_1(\kappa r')\tilde{f}_u(r'), \;\; g_2(r) = -\int_{r_0}^r dr'\, \chi^{\mathrm{ANO}}(r') I_1(\kappa r')\tilde{f}_u(r'). \tag{60}$$

Here I_1 is the modified Bessel function [10]. The constant b_0 will be determined from matching conditions at $r = r_0$. We see that for very large r, i.e. for $r \gg k_0^{-1}$,

$$\tilde{h}_0(r) \sim \sqrt{r} \exp(-\kappa r).$$

This component of the backreaction on the vector field is relatively uninteresting. It gives a small, time- and ξ^3-independent, correction to the $\chi^{\rm ANO}(r)$ field of the same range as $\chi^{\rm ANO}$ (the penetration depth $\sim \kappa^{-1} m_H^{-1}$).

Much more interesting are the other components, i.e. the ones with $\Omega_\pm = \pm 2\omega_0$. In this case, Eq. (57) reduces in the outer region to

$$\tilde{h}''_\pm - \frac{1}{r}\tilde{h}'_\pm + (4\omega_0^2 - \kappa^2)\tilde{h}_\pm = \chi^{\rm ANO}(r)\tilde{f}_u(r), \tag{61}$$

where again $\chi^{\rm ANO}$ is given by formula (32). Applying the standard methods [11], we obtain the solution

$$\begin{aligned}\tilde{h}_\pm = r h_1(r) H_1^{(1)}(\sqrt{4\omega_0^2 - \kappa^2}\; r) + r h_2(r) H_1^{(2)}(\sqrt{4\omega_0^2 - \kappa^2}\; r) \\ + b_\pm \sqrt{4\omega_0^2 - \kappa^2}\; r H_1^{(1,2)}(\sqrt{4\omega_0^2 - \kappa^2}\; r),\end{aligned} \tag{62}$$

where $H_1^{(1)}, H_1^{(2)} = (H_1^{(1)})^*$ are Hankel functions, and

$$h_1(r) = \frac{i\pi}{4}\int_r^\infty dr'\; H_1^{(2)}(\sqrt{4\omega_0^2 - \kappa^2}\; r')\chi^{\rm ANO}(r')\tilde{f}_u(r'), \tag{63}$$

$$h_2(r) = (h_1(r))^*,$$

In order to keep χ real-valued we have to assume that $b_+ = (b_-)^*$. In the last term on the r.h.s. of formula (62) one should take $H_1^{(1)}(H_1^{(2)})$ in the positive (negative) frequency component in order to satisfy the outgoing radiation condition.

The functions $h_{1,2}$ exponentially vanish for large r. On the other hand, for large z [10]

$$H^{(1)}(z) \cong \sqrt{\frac{2}{\pi z}} \exp(i(z - 3\pi/4)), \tag{64}$$

hence the last term on the r.h.s. of formula (62) grows like $r^{1/2}$. It gives the radiation of the vector field from the vortex.

In the small r region ($r \le r_0$), for the all frequencies, we shall be satisfied with approximate polynomial solutions in the form

$$\underline{h}_{0,+} = \frac{1}{2}d_{0,+}r^2 + \frac{3f_1}{16r_0}r^4 + \frac{1}{16r_0}(f_3 - \frac{f_1}{16})r^6, \tag{65}$$

$$\underline{h}_- = (\underline{h}_+)^*.$$

They have been obtained by inserting the formulae (29, 30, 52) in Eq. (57), and by neglecting all terms $\sim \kappa^2$ on the l.h.s. of it (let us remind that we consider the case $\kappa \ll 1/2$). Again, we have taken a polynomial of the maximal order compatible with the fact that for small r the r.h.s. of Eq.(57) is known up to terms of the order r^4.

The constants $b_0, b_\pm, d_0, d_\pm$ are determined from matching conditions at $r = r_0$:

$$\underline{h}_{0,\pm}(r_0) = \tilde{h}_{0,\pm}(r_0), \quad \underline{h}'_{0,\pm}(r_0) = \tilde{h}'_{0,\pm}(r_0).$$

It turns out that the first two terms on the r.h.s. of formula (62) computed for $r = r_0$ are of the order $\kappa \ln(\kappa)$ and therefore they can be neglected for small κ. The argument of the Hankel function $H_1^{(1)}$ is very small for $r = r_0$, so we may use the following approximate formula [10]

$$H_1^{(1)}(z) \approx -i0.64/z.$$

Similarly,

$$\kappa r_0 K_1(\kappa r_0) \approx 1.$$

The resulting from the matching conditions sets of linear algebraic equations, simplified as described above, give

$$d_0 \approx \kappa g_1(r_0) - \frac{1}{2} f_1 r_0 - \frac{3}{8}, \quad b_0 \approx -\frac{5}{9} f_1 r_0 - \frac{2}{3}, \tag{66}$$

$$d_\pm \approx -\frac{8}{3} f_1 r_0^{-1} - 2 r_0^{-2}, \tag{67}$$

and

$$b_\pm \approx \pm 1.56 i b_0 \mp \frac{1}{2} i\pi \kappa^{-1} (4\omega_0^2 - \kappa^2)^{-1/2} C(\kappa), \tag{68}$$

where $C(\kappa)$ is a constant given by the following formula

$$C(\kappa) = \int_{\kappa r_0}^{\infty} ds\ J_1(\sqrt{3 - 4k_0^2 \kappa^{-2}}\ s) s K_1(s) \tilde{f}_u(s/\kappa).$$

In formula (68) only the second term on the r.h.s. is important because κ^{-1} is very large. The constants r_0 and f_1 we know from Sections 3 and 4. This completes the determination of the backreaction on the vector field.

The most interesting implication of these formulae is that the backreaction includes radiation from the excited vortex. The $b_\pm$ components of $\tilde{h}_\pm$ inserted in formulae (56), (39) give the following contribution to $\chi^{(2)}$ for $r \gg r_0$

$$\delta\tilde{\chi}^{(2)} \approx -\kappa \omega_0^2 (4\omega_0^2 - \kappa^2)^{-1/4} N^2 \sqrt{\pi/2} C(\kappa) \sqrt{r} \cos(-2E(q)\xi^0 + 2q\xi^3 + \sqrt{4\omega_0^2 - \kappa^2}\ r - \pi/4), \tag{69}$$

where $E(q) = \sqrt{\omega_0^2 + q^2}$. This radiation wave has a conical shape – at fixed time the surface of constant phase, determined by the following condition

$$2q\xi^3 + \sqrt{4\omega_0^2 - \kappa^2}\ r = \pi/4 + 2E(q)\xi^0 + const,$$

is a cone with its top point lying on the ξ^3 axis. For $q = 0$ the cone degenerates to a cylinder of constant r. Opening angle γ_0 of the cone, given by the formula

$$\tan\gamma_0 = \frac{2q}{\sqrt{4\omega_0^2 - \kappa^2}},$$

increases with the wave number q of the Proca wave on the vortex. The wave four-vector corresponding to the radiation wave (69) has the following components

$$(k^\mu) = \left(2E(q), \vec{k}\right),$$

where

$$\vec{k} = \sqrt{4\omega_0^2 - \kappa^2}\left(r^1/r,\ r^2/r,\ \tan\gamma_0\right).$$

Its 4-dimensional invariant length is equal to κ^2, in accordance with the fact that the vector particle has the mass equal to κ in the m_H units.

The energy flux related to that conical wave is given by the Poynting vector

$$\vec{S} = \delta\vec{E} \times \delta\vec{B},$$

where $\delta\vec{E}, \delta\vec{B}$ are the contributions to electric and magnetic fields corresponding to $\delta\chi^{(2)}$. Simple calculation gives

$$\vec{S} = \pi C(\kappa)^2 \kappa^2 \omega_0^4 (4\omega_0^2 - \kappa^2)^{-1/2} N^4 m_H^4 E(q) \frac{1}{r} \sin^2(k^\mu x_\mu m_H)\vec{k}. \tag{70}$$

5.2 The wave packet case

Calculations of backreaction of the Proca wave packet on the vector field χ are carried out in the same steps as in the plane wave case. Instead of formula (56) we have now

$$h = -2(2\pi)^{1/2}\kappa^2 \Big[\tilde{w}_1(k)\delta(\omega)h_0(r) + \frac{1}{2}\tilde{w}_2(k)\delta(\omega - 2\omega_0)h_+(r) + \frac{1}{2}\tilde{w}_2(k)\delta(\omega + 2\omega_0)h_-(r)\Big]. \tag{71}$$

Equations for the frequency components $h_0,\ h_\pm$ coincide with Eq. (57), where Ω_a^2 is replaced by $-k^2$ or $4\omega_0^2 - k^2$. However, taking into account the assumption that the wave packet (24) has momentum cutoff at $k \sim \Lambda^{-1}$ where $\omega_0\Lambda \gg 1$, we see that for our wave packet $k^2 \ll \omega_0^2$ so that we may neglect the $-k^2$ term altogether. Hence, we again obtain Eq.(57) with $\Omega_0^2 = 0$, $\Omega_\pm^2 = 4\omega_0^2$ as in the plane wave case. Therefore, the discussion following Eq. (57) applies also to the present case, including results (66-68) for constants $d_\pm,\ c_\pm$. We again find the radiation. Using formula (62) and asymptotic formula (64) for the Hankel function we obtain the following form of the radiation part of the backreaction in the standing wave packet case

$$\delta\tilde{\chi}^{(2)} \approx (2\pi)^{1/2}\kappa(4\omega_0^2 - \kappa^2)^{-1/4} C(\kappa) w_2(\xi_3)\sqrt{r}\cos(-2\omega_0\xi^0 + 2q\xi^3 + \sqrt{4\omega_0^2 - \kappa^2}\, r - \pi/4), \tag{72}$$

It is a cylindrical wave, modulated along the vortex by the $w_2(\xi_3)$ function. The corresponding wave vector has the components

$$(k^\mu) = \left(2\omega_0, \sqrt{4\omega_0^2 - \kappa^2}\; r^1/r, \sqrt{4\omega_0^2 - \kappa^2}\; r^2/r, 0\right).$$

Its 4-dimensional invariant length is equal to κ^2.

The fact that the wave front has the cylindrical shape is in accordance with the conical character of the emitted wave in the previous example. The standing wave packet can be regarded as containing $+q$ and $-q$ components with an equal weight – such a superposition of conical waves (69) gives the cylindrical wave.

Proca wave packets moving along the vortex can be obtained by Lorentz boosts.

6 REMARKS

1. Let us summarize the main points of our paper. In order to investigate the excited vortex fields, introduced by the Ansatz (2-4), we have used the perturbative expansion in the amplitude of the excitation. The backreaction is given as the corrections $F^{(2)}$, $\chi^{(2)}$ to the unexcited vortex fields F^{ANO}, χ^{ANO}. We have found that $F^{(2)}$ exponentially vanishes, but the corresponding characteristic length is large ($\sim k_0^{-1} m_H^{-1}$), much larger than the coherence length ($\sim m_H^{-1}$) and the penetration depth ($\sim \kappa^{-1} m_H^{-1}$). For the very small values of κ assumed in our paper, there is no radiation of the scalar field from the excited vortex. The vortex fields F, χ couple to $W_\alpha W^\alpha$ – this composite field has an effective mass equal to $4\omega_0^2$. For $\kappa \ll 1/2$ this characteristc frequency turns out to be too small to induce radiation of the Higgs field F, but it is large enough for radiation of the vector field χ. Nevertheless, radiation of the Higgs field might appear in higher orders in the perturbative expansion. The frequency of the radiative component of $\delta\tilde{\chi}^{(2)}$, c.f. formulae (69), (72), is twice of that of the excitation field $W_\alpha(\xi)$.

2. Perhaps the main question about our approach is whether the approximations used are correct. Rigorous convergence proofs are not known neither for the polynomial approximation nor for the expansion in the amplitude of the excitation. Confrontation of the polynomial approximation for the unexcited ANO vortex with a purely numerical solution shows that it works rather well [12]. One reason is that the polynomials are used in a finite interval of r ($0 \leq r \leq r_0$), where the vortex fields are expected to have a simple behaviour (e.g., one does not expect that they oscillate like $r\sin(1/r)$ for $r \to 0$). Therefore, low order polynomials can give quite good approximation.

Another reason is that the scalar and vector fields in the Abelian Higgs model are massive ones, and therefore the vortex fields approach the corresponding vacuum values very rapidly (exponentially). Therefore, deviations from the vacuum values for large r can be regarded as small corrections which one may reliably determine from linearised equations.

As for the expansion in powers of the amplitude, certainly encouraging is the fact that the correction $F^{(2)}$ has turned out to be small for all values of r. So is $\chi^{(2)}/r$ (which is the function actually present in our formulae). Moreover, the exact equations (6,8,9,10) depend on the excitation fields A^{β}, ϑ in a smooth manner, and the corresponding terms are just additional contributions to the other nonlinear terms which are already present in the case of the unexcited ANO vortex. The presence of the excitation fields A_{ϑ}, ϑ in Eq. (6) does not seem to introduce any singularity in that equation. All that suggests that also the solution describing the excited vortex depends smoothly on the amplitude of the excitation field if the amplitude is small.

3. The presence of radiation means that the excitation is not stable. Nevertheless, lifetime of the small amplitude excitations is large because energy of the excitation is proportional to N^2 (for concreteness we are referring to the plane wave excitation) while the energy flux is of the order N^4. Our formulae also show that the energy flux increases when the amplitude grows, however our approximations should not to be trusted when the amplitude becomes large.

4. Finally, let us mention two directions for rather interesting extensions of the present work. According to our results, the excited vortex is no longer localised within the penetration depth range $((\kappa m_H)^{-1})$. This likely influences its interaction with other vortices. The new interaction, unknown as yet, may turn out to have quite interesting properties.

Also, it would be interesting to apply the polynomial approximation in order to calculate radiation of Goldstone particles from global vortices. Such an analytical approach would nicely complement existing numerical investigations as well as analytical results based on Lund-Regge string model for the global vortices presented in, e.g., [13], where one can also find references to older papers on radiation from the global vortices.

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